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COUNTING THE RELATIVELY FINITE FACTORS OF A BERNOULLI SHIFT

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ABSTRACT

Let T acting on $(\Omega, \mathcal{F}, \mu)$ be a finite entropy Bernoulli shift. A T invariant factor $\mathfrak{A} \subset \mathcal{F}$ is "relatively finite" if a.e. fiber of $\mathfrak{A}$ has a finite, hence constant k, number of points. We say two factors $\mathfrak{A}, \mathfrak{A}' \subset \mathcal{F}$ "sit the same" if there is a measurable measure preserving map φ with $\varphi T \varphi^{-1} = T$ and $\varphi(\mathfrak{A}) = \mathfrak{A}'$. We show here that up to sitting the same there are only finitely many relatively finite factors with k point fibers in a Bernoulli shift, and that they are classified by a certain algebraic structure in the symmetric group on k-points.

In [1] Ornstein introduced the notion of the shift invariant factors, A, A' of a measure preserving transformation $(T, \Omega, \mathcal{F}, \mu)$ "sitting the same". This is true if there is an automorphism φ of T with $\varphi(A) = A'$. There he presented the idea that the theory of factors of a Bernoulli shift T, up to the equivalence of sitting the same should parallel the isomorphism theory for measure preserving transformations. In keeping with this idea, one can define what it means for T to be A-relatively weak mixing, A-relatively mixing, A-relatively K and A-relatively Bernoulli. M. Rahe, [2], in the appendix to his Ph.D. dissertation has shown that the basic structure of K-automorphisms is mirrored in factors A where T is A-relatively K. J.-P. Thouvenout's [7] theory of factors A which have an invariant Bernoulli compliment can be read as the translation of Ornstein's theory of Bernoulli shifts to factors A where T is A-relatively Bernoulli. What we wish to do here is to translate to the theory of factors of a Bernoulli shift the structure of the simplest of transformations, those on a finite point set. We say T has A-relatively k points, if the factor A has k point fibers. If the parallel holds true, for any k, a Bernoulli shift T should have only finitely many factors with k point fibers, up to sitting the same, and these should be

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characterized by some very simple structure. This is precisely what we shall show. This result fails for T anything less than a Bernoulli shift.

The basic tools for our argument are the structure theorems in [3], [4], and [5], but not the arguments which lead to their proof. Thus no more than a passing acquaintance with these arguments is assumed. With these results our argument becomes very easy.

Suppose we have a factor A of a Bernoulli shift with k point fibers. We can, then, write $\Omega \cong \bar{\Omega} \times \{1, \cdots, k\}$, $\mathcal{F} \cong A \times \{1, \cdots, k\}$, and

$$T(\omega) \cong T_1(\bar{\omega}, i) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}}(i)), \tag{1.1}$$

where $(\bar{T}, \bar{\Omega}, a, \bar{\mu})$ is a realization as a point map of T restricted to A, and $g_{\bar{\omega}}; \bar{\Omega} \to s(k)$, the symmetric group on k points. The first coordinate is the factor A and the second is a labeling for the k points of the fiber. A map T, defined by $\bar{T}$ and $g_{\bar{\omega}}$ as in (1.1) we will call a k-point extension of $\bar{T}$.

Suppose we have two factors A, A' with k point fibers. We can write T, then, in two ways as a k-point extension,

$$T_1(\bar{\omega}, i) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}}(i))$$

and

$$T_2(\bar{\omega}', i) = (\bar{T}'(\bar{\omega}'), g'_{\bar{\omega}'}(i)).$$

In this context A and A' sit the same iff there is a map $\varphi; \bar{\Omega} \to \bar{\Omega}'$ and an $\alpha_{\bar{\omega}}; \bar{\Omega} \to s(k)$ so that

$$\varphi \bar{T} \varphi^{-1} = \bar{T}' \tag{1.2}$$

and

$$g_{\varphi(\bar{\omega})} = \alpha_{\bar{T}(\bar{\omega})} \circ g_{\bar{\omega}} \circ \alpha_{\bar{\omega}}^{-1}.$$

If (1.2) is satisfied the automorphism $(\bar{\omega}, i) \to (\varphi(\bar{\omega}), \alpha_{\bar{\omega}}(i))$ takes T_1 to T_2 and A to A', and if an automorphism exists, φ is its restriction taking $A \to A'$, and $\alpha_{\bar{\omega}}$ is the relabeling function on the fibers.

What we will consider from now on are k-point extensions of some $\bar{T}$ of the form

$$T(\omega, i) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}}(i)) \tag{1.3}$$

and will say two such are "factor isomorphic" if there is a φ and $\alpha_{\bar{\omega}}$ satisfying (1.2). We will first characterize k-point extensions of a Bernoulli $\bar{T}$ up to factor isomorphism, and then discuss when the k-point extensions are themselves Bernoulli.

For a T defined by (1.3) we can define its "full extension" $\hat{T}$; $\bar{\Omega} \times s(k) \to \bar{\Omega} \times s(k)$ by

$$\hat{T}(\bar{\omega}, g) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ g), \tag{1.4}$$

i.e. use the same skewing function $g_{\bar{\omega}}$ but replace $\{1, \cdots, k\}$ by $s(k)$, and act by left multiplication. Let H be the subgroup of $s(k)$ which fixes $\{1\}$. Its left cosets in $s(k)$ are $H_1, H_2, \cdots, H_k$ where H_i is the set of all permutations taking $1 \to i$. Now $\hat{T}$ acts on the fibers $(\bar{\omega}, H_i)$ by $\hat{T}(\bar{\omega}, H_i) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ H_i) = (\bar{T}(\bar{\omega}), H_{g_{\bar{\omega}}(i)})$. Mapping $((\bar{\omega}), H_i)$ to $(\bar{\omega}, i)$ we see that T is a factor of $\hat{T}$ with $(k-1)!$ point fibers.

We can now make $\hat{T}$ part of a larger group action. For any $\bar{g} \in s(k)$, write

$$\hat{T}_{\bar{g}}(\bar{\omega}, g) = (\bar{\omega}, g \circ \bar{g}). \tag{1.5}$$

Now $\hat{T} \circ \hat{T}_g = \hat{T}_g \circ \hat{T}$, and what we have is $Z \times s(k)$ action $\{\hat{T}_g\}_{g \in Z \times s(k)}$. Our first lemma explains the introduction of $\hat{T}$.

Lemma 1. *Suppose T and S are two k-point extensions*

$$T(\bar{\omega}, i) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}}(i))$$

and

$$S(\bar{\omega}', i) = (\bar{S}(\bar{\omega}), g'_{\bar{\omega}}(i)).$$

T and S are factor isomorphic iff the $Z \times s(k)$ actions $\{\hat{T}_g\}_{g \in z \times s(k)}$ and $\{\hat{S}_g\}_{g \in z \times s(k)}$ are isomorphic.

Proof. Suppose the two $z \times s(k)$ actions are isomorphic by a map $\hat{\varphi}$. Now $\hat{\varphi}$ must take the algebra of $s(k)$ invariant sets for $\hat{T}_g$ to the algebra of $s(k)$ invariant sets for $\hat{S}_g$. These are the corresponding first coordinates $\bar{\Omega}$ and $\bar{\Omega}'$. Let $\bar{\varphi}$ be the restriction of $\hat{\varphi}$ to these algebras. Now $\bar{\varphi}\bar{T}\bar{\varphi}^{-1} = \bar{S}$. Let $\alpha_{\bar{\omega}}$ be the relabeling function on the $s(k)$ fibers. As $\hat{\varphi}$ takes the $s(k)$ action on the fiber over $\bar{\omega}$ to the $s(k)$ action on the fiber over $\varphi(\bar{\omega})$, $\alpha_{\bar{\omega}}$ must be left multiplication by $\alpha_{\bar{\omega}}(e) = \bar{\alpha}_{\bar{\omega}}$. Thus

$$\hat{\varphi}(\bar{\omega}, g) = (\bar{\varphi}(\bar{\omega}), \bar{\alpha}_{\bar{\omega}} \circ g), \tag{1.6}$$

and so

$$\bar{\varphi}\bar{T}\bar{\varphi}^{-1} = \bar{S} \qquad \text{and} \qquad g'_{\bar{\varphi}(\bar{\omega})} = \bar{\alpha}_{\bar{T}(\bar{\omega})} \circ g_{\bar{\omega}} \circ \bar{\alpha}_{\bar{\omega}}^{-1}. \tag{1.7}$$

As (1.7) and (1.2) are identical, the result follows. □

We have now modified the characterization of k-point extensions up to factor isomorphism to characterizing $Z \times s(k)$ actions up to isomorphism.

Already we can begin to see how theorems 1 and 3 of [5] will help us, as they say if $\hat{T}$ and $\hat{S}$ are both Bernoulli, then the whole $Z \times s(k)$ actions must be isomorphic and hence T and S are factor isomorphic. This already handles the case of $k = 2$.

COROLLARY 1. *In a Bernoulli shift T any two factors with two-point fibers sit the same.*

PROOF. Take two such, and write T in two ways, T_1 and T_2, as a two point extension. Now $T \cong T_1 \cong \hat{T}_1$ and $T \cong T_2 \cong \hat{T}_2$, and hence both $\hat{T}_1$ and $\hat{T}_2$ are Bernoulli of the same entropy. The result follows from theorems 1 and 3 of [5], and Lemma 1. □

Some examples due to P. Shields show that the situation is already more complicated for $k = 3$. Let $\bar{T}$ be the independent process $(1/3, 1/3, 1/3)$ with generator $\{a, b, c\}$. Define

$$g^1_{\bar{\omega}} = \begin{cases} (0,1,2) & \text{if } \bar{\omega} \in a \\ (0)(1)(2) & \text{if } \bar{\omega} \in b, c \end{cases}$$

$$g^2_{\bar{\omega}} = \begin{cases} (0,1)(2) & \text{if } \bar{\omega} \in a \\ (0)(1,2) & \text{if } \bar{\omega} \in b, c \end{cases}$$

$$g^3_{\bar{\omega}} = \begin{cases} (0,1,2) & \text{if } \bar{\omega} \in a \\ (0,1)(2) & \text{if } \bar{\omega} \in b \\ (0)(1)(2) & \text{if } \bar{\omega} \in c \end{cases},$$

and

$$T_j(\bar{\omega}, i) = (\bar{T}(\bar{\omega}), g^j_{\bar{\omega}}(i)). \tag{1.8}$$

It is easy to check that all T_j, on $\{a, b, c\} \vee \{1, 2, 3\}$ are mixing Markov processes, hence isomorphic to Bernoulli shifts. Thus all three first coordinates can be thought of as factors with three point fibers of this Bernoulli shift.

It is also easy to check, though, that $\hat{T}_1$ is nonergodic, $\hat{T}_2$ is ergodic but $\hat{T}_2^2$ is not, and $\hat{T}_3$ on $\{a, b, c\} \vee S(3)$ is mixing Markov. Thus none are isomorphic and the three factors must sit differently.

Notice that the main result of [4] tells us that, in fact, the only way $\hat{T}_j$ could be non-Bernoulli is by having a nonergodic power. This will be the starting point for our analyses of $\hat{T}$.

Suppose $\hat{T}$ is the full extension of some k-point extension of a Bernoulli shift $\bar{T}$, i.e.

$$\hat{T}(\bar{\omega}, g) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ g).$$

By corollary 15 of [4], if $\hat{T}$ is not Bernoulli, then it must have a Pinsker algebra made of a finite number of atoms. Let $S^1_1, S^1_2 \cdots S^1_{l(1)}, S^2_2 \cdots S^2_{l(2)}, \cdots S^t_1 \cdots S^t_{l(t)}$ be the atoms of this algebra where $\hat{T}(S^j_i) = S^j_{(i+1) \bmod l(i)}$. Furthermore, as $\bar{T}$ is Bernoulli, for a.e. $\bar{\omega}, S^j_i \cap (\bar{\omega}, s(k)) = S^j_i(\bar{\omega})$ has a fixed number of points in it, independent of $\bar{\omega}$.

For any $\bar{g} \in s(k)$

$$\hat{T}_{\bar{g}} \bigcup_{i=1}^{l(j)} S^j_i = \bigcup_{i=1}^{l(j')} S^{j'}_i \tag{1.9}$$

for some j', as these are the ergodic components of $\hat{T}$. Further

$$\hat{T}_{\bar{g}}(S^j_i) = S^{j'}_{(i_0+i) \bmod l(j')} \qquad \text{for all } i = 1, \cdots, l(j) \tag{2.0}$$

as $\hat{T}$ moves both sides cyclically.

A $\hat{T}_{\bar{g}}$ can be found taking any S^j_i to any other. Hence $l(j) = l$, a constant, and $\mathrm{card}(S^j_i(\bar{\omega})) = \alpha$, independent of i, j and $\bar{\omega}$, and $k! = tl\alpha$.

Let $H^j_i \subset s(k)$ be the subgroup such that $\hat{T}_h(S^j_i) = S^j_i$ for $h \in H^j_i$, i.e. the subgroup which fixes this atom of $\Pi(\hat{T})$. Let $\bar{g}$ be any element with $T_{\bar{g}}(S^j_i) = S^{j'}_{i'}$. It follows that $H^{j'}_{i'} = \bar{g} H^j_i \bar{g}^{-1}$, so any two such subgroups differ by an inner automorphism.

Pick $H = H^1_1$. Now notice that for $h \in H$, $S^1_1(\bar{\omega}) \circ h = S^1_1(\bar{\omega})$, i.e. $S^1_1(\bar{\omega})$ is a left coset of H. We would like it to be precisely H, for all $\bar{\omega}$. To achieve this we will relabel the $s(k)$ coordinate as follows. Let $\gamma(\bar{\omega}) \in S^1_1(\bar{\omega})$, a choice for coset representative. And now define

$$\psi(\bar{\omega}, g) = (\bar{\omega}, \gamma^{-1}(\bar{\omega}) \circ g), \tag{2.1}$$

and

$$\hat{T}'_g = \psi \hat{T}_g \psi^{-1}.$$

Surely $\{\hat{T}'_g\} \cong \{\hat{T}_g\}$, and $\hat{T}'_h$ leaves $\psi(S^1_1)$ invariant, for $h \in H$. Now $\psi(S^1_1)$ is precisely $\bar{\Omega} \times H$. For any other $H^j_i = \bar{g}^{-1} H \bar{g}$, the set left invariant is $\psi(S^j_i) = \bar{\Omega} \times H\bar{g}$. Thus, after this relabeling, the atoms of $\Pi(\hat{T}')$ can be identified as right cosets of a subgroup H, where the choice for H is unique up to an inner automorphism. Now

$$\hat{T}'(\bar{\omega}, H) = (\bar{T}(\bar{\omega}), g'_{\bar{\omega}} H)$$

where

$$g'_{\bar{\omega}} = \gamma^{-1}(\bar{\omega}) g_{\bar{\omega}} \gamma(\bar{\omega}). \tag{2.2}$$

But now for a.e. $\bar{\omega}$

$$g'_{\bar\omega}H = H\bar{g}_0 \quad \text{for some } \bar{g}_0, \tag{2.3}$$

by (20). Thus $\bar{g}_0^{-1}H\bar{g}_0 = \bar{g}_0^{-1}g'_{\bar\omega}H$, and so

$$\bar{g}_0^{-1}H\bar{g}_0 = H. \tag{2.4}$$

Now we know that

$$\hat{T}(\bar{\Omega} \times H) = \bar{\Omega} \times H\bar{g}_0 \tag{2.5}$$

where $\bar{g}_0$ is in $N(H)$, the normalizer of H in $s(k)$. Once H is fixed, our choice for $\bar{g}_0$ is unique up to multiplication by an element of H.

Notice

$$\hat{T}'^2(\bar\omega, H) = \hat{T}'(\bar{T}(\bar\omega), H\bar{g}_0) = (\bar{T}^2(\bar\omega), g'_{T(\omega)}H\bar{g}_0) = (\bar{T}^2(\bar\omega), H\bar{g}_0^2)$$

and in general

$$\hat{T}'^i(\bar{\Omega} \times H) = \bar{\Omega} \times H\bar{g}_0^i, \tag{2.6}$$

and for any other coset

$$\hat{T}'^i(\bar{\Omega} \times H\bar{g}) = \bar{\Omega} \times H\bar{g}_0^i\bar{g} = \bar{g}_0^iH\bar{g}.$$

Now we know precisely how the atoms of $\Pi(\hat{T}')$ cycle under the action of $\hat{T}'$, they rotate under left multiplication by $\bar{g}_0$.

We now want to show that knowing H and $\bar{g}_0 \in N(H)$ characterizes the $Z \times S(k)$ action. To do this, define

$$\tilde{T}_1 = \hat{T}'_{\bar{g}_0^{-1}} \circ \hat{T}'/\bar{\Omega} \times H. \tag{2.7}$$

This is well defined as $\hat{T}'$ takes $\bar{\Omega} \times H$ to $\bar{\Omega} \times H\bar{g}_0$, which $\hat{T}'_{\bar{g}_0^{-1}}$ takes to $\bar{\Omega} \times H$. Also, for $h \in H$,

$$\tilde{T}_h = \hat{T}'_h/\bar{\Omega} \times H. \tag{2.8}$$

Now as $\tilde{T}_1 \circ \tilde{T}_h = \tilde{T}_{g^{-1}hg_0}\tilde{T}_1$, these define a $z^{g\delta} \otimes H$ action (see [5] for more discussion), we can apply the isomorphism theory of [5] if we know $\tilde{T}_1$ to be Bernoulli. Let v be the order of $\bar{g}_0$. It follows that $\tilde{T}_1^v = \hat{T}'^v/\bar{\Omega} \times H$. As $\bar{\Omega} \times H$ is an atom of $\Pi(\hat{T}') = \Pi(\hat{T}'^v)$, $\hat{T}'^v/\bar{\Omega} \times H$, by Corollary 15 of [4] must be Bernoulli, and so $\tilde{T}_1$ is also.

We can now state a lemma.

LEMMA 2. *If we have two k-point extensions of a Bernoulli shift, S and T, and through the above analyses obtain the same H and $\bar{g}_0 \in N(H)$ for each, then S and T are factor isomorphic.*

PROOF. Let $\{\tilde{S}_g\}_{g\in z\otimes^{\delta\delta}H}$ and $\{\tilde{T}_g\}_{g\in z\otimes^{\delta\delta}H}$ be the corresponding $z\otimes^{\delta\delta}H$ actions. Now $\tilde{S}_1$ and $\tilde{T}_1$ are both Bernoulli shifts, of the same entropy. By theorems 1 and 3 of [5], these two group actions are isomorphic. Hence there is a $\varphi;\bar{\Omega}\to\bar{\Omega}'$ and $\alpha_{\bar{\omega}};\bar{\Omega}\to H$ so that if we define

$$\psi(\bar{\omega},h)=(\varphi(\bar{\omega}),\alpha_{\bar{\omega}}\circ h),$$

then

$$\psi\tilde{T}_h\psi^{-1}=\tilde{S}_h,\qquad h\in z\otimes^{\delta\delta}H.$$

Define $\hat{\psi}(\bar{\omega},g)=(\varphi(\bar{\omega}),\alpha_{\bar{\omega}}\circ g)$, $(\bar{\omega},g)\in\bar{\Omega}\times s(k)$. We certainly get

$$\hat{\psi}(\hat{T}'_g)\hat{\psi}^{-1}=\hat{S}'_g\quad\text{for}\quad \bar{g}\in s(k).$$

Thus we only need to check $\hat{\psi}\hat{T}'\hat{\psi}^{-1}=\hat{S}^{-1}$ on $\Omega\times e$ to have $\hat{\psi}$ an isomorphism of the $z\times s(k)$ actions. Now

$$\begin{aligned}\hat{\psi}\hat{T}'\hat{\psi}^{-1}(\bar{\omega},e)&=\hat{S}'_{\bar{g}_0}(\hat{\psi}\hat{T}_{\bar{g}_0^{-1}}\circ\hat{T}'\psi^{-1})(\bar{\omega},e)=\hat{S}'_{\bar{g}_0}(\psi\tilde{T}_1\psi^{-1})(\bar{\omega},e)\\&=\hat{S}'_{\bar{g}_0}\tilde{S}_1(\bar{\omega},e)=\hat{S}'_1(\bar{\omega},e)\end{aligned}$$

and $\hat{\psi}$ is an isomorphism. Lemma 1 completes the result. □

Thus if in our analyses we make the same choices for H and $\bar{g}_0\in N(H)$, the extensions are factor isomorphic.

On the other hand, our choice for H is unique up to inner automorphisms, so if our choices did not differ by an inner automorphism the factors would not sit the same. If they did, we could change our choices to be the same. If our choices for H are the same, but those for $\bar{g}_0$ do not lie in the same coset of H in $N(H)$, then the factors cannot sit the same, but if they do we can make our choices the same.

In line with this, let $\mathcal{H}_i$, $i=1,\cdots,n(k)$ be the equivalence classes of subgroups $H\subset s(k)$ where two are equivalent if they differ by an inner automorphism, and let $m(i)$, $i=1,\cdots,n(k)$ be $|N(H)|/|H|$, $H\in\mathcal{H}_i$. What we have now shown is the following

COROLLARY 2. *The number of k-point extensions of a Bernoulli shift, up to factor isomorphism, is at most* $\sum_{i=1}^{n(k)}m(i)$. □

It will be $=\sum_{i=1}^{n(k)}m(i)$ if we can get an example for every H and $\bar{g}_0\in N(H)$. Do this as follows.

Let $H=\{h_1,\cdots,h_l\}$. Pick any Bernoulli shift $\bar{T}$, with an independent generator $P=\{P_1,\cdots,P_{l'}\}$, $l'\geqq l$. Now define

$$g_{\bar{\omega}} = \begin{cases} \bar{g}_0 \circ h_i & \text{if } \bar{\omega} \in P_i, \quad i \leqq l \\ \bar{g}_0 & \text{if } \bar{\omega} \in P_i, \quad i > l \end{cases} \tag{2.9}$$

and

$$T(\bar{\omega}, g) = (\bar{T}(\bar{\omega}), g_{\bar{\omega}} \circ g). \tag{3.0}$$

All we need to check is that the atoms of $\Pi(T)$ are the $\bar{\Omega} \times Hg$. This is true if $T^v / \bar{\Omega} \times H$ is Bernoulli, v the order of $\bar{g}_0$. But $T^v / \bar{\Omega} \times H$ on $\bigvee_{i=1}^{v} T^i(P) \vee H / \bar{\Omega} \times H$ is v-step mixing Markov. Thus for this H and $\bar{g}_0 \in N(H)$ we have an example.

THEOREM 1. *The number of* k*-point extensions of a Bernoulli shift, up to factor isomorphism is* $\sum_{i=1}^{n(k)} m(i)$. □

We now want to count the factors of a Bernoulli shift. This will be the number of k-point extensions which are themselves Bernoulli. Our last result gives this.

THEOREM 2. *The number of* k*-point factors of a Bernoulli shift, up to sitting the same, is* $\sum_{i \in I} m(i)$ *where* $i \in I$ *iff the subgroups in* $\mathcal{H}_i$ *are transitive on* $\{1, \cdots, k\}$.

PROOF. For a fixed H and $\bar{g}_0 \in N(H)$, take the example given in (2.9) and (3.0). If H is not transitive, then T^v will not be ergodic, leaving invariant the sets of intransitivity in the second coordinate. On the other hand if H is transitive on $1, \cdots, k$, then T on $P \vee \{1, \cdots, k\}$ is mixing Markov, hence Bernoulli. Thus those k-point extensions which give Bernoulli shifts are precisely those where H is transitive. □

This completes our argument. Notice that the three 3-point examples of P. Shields are all of them. It would be interesting to know something of how the numbers

$$L(k) = \sum_{i=1}^{n(k)} m(i) \quad \text{and} \quad L'(k) = \sum_{i \in I} m(i) \quad \text{behave in } k.$$

In [6] one gets two factors with two-point fibers in a K-automorphism which do not sit the same. This easily extends to uncountably many. They do not sit the same because they are nonisomorphic. It is possible, though, to have them not sit the same even when they are isomorphic. This example is part of a general technique for constructing examples we will develop in a later paper. These examples show how badly the arguments here fail outside the class of Bernoulli shifts.

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